MMAT 5010 Linear Analysis (2020-21): Homework 1. Deadline: 25 Jan 2021

Important Notice:

♣ The answer paper must be submitted before the deadline.
♠ The answer paper MUST BE sent to the CU Blackboard. Please refer to the course web for details.

1. Let X be a normed space. Show that X is a Banach space if and only if the unit sphere $S := \{x \in X : \|x\| = 1\}$ of X is complete, that is, every Cauchy sequence (x_n) in S has the limit in S.

2. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be the normed spaces over the field $\mathbb{K}$. The direct sum of X and Y, write $X \oplus Y$, is defined by $X \oplus Y := \{(x, y) : x \in X, y \in Y\}$ under the addition and the scalar multiplication: $(x, y) + (x', y') := (x + x', y + y')$ and $t(x, y) := (tx, ty)$ for $(x, y); (x', y') \in X \oplus Y$ and $t \in \mathbb{K}$.
For each $(x, y) \in X \oplus Y$, let

$$q(x, y) := \|x\|_X + \|y\|_Y.$$

   (a) Show that q is a norm function on $X \oplus Y$.

   (b) Show that $X \oplus Y$ is a Banach space under the norm q if and only if X and Y both are Banach spaces.

$* * *$ **End** $* * *$